

Koszulity for skew PBW extensions over fields

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Abstract

Koszul and homogeneous Koszul algebras were defined by Priddy in [18]. There exist some relations between these algebras and the skew PBW extensions introduced in [8]. In this paper we give conditions to guarantee that skew PBW extensions over fields are Koszul or homogeneous Koszul. We also show that a constant skew PBW extension of a field is a PBW deformation of its homogeneous version.

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1 Introduction

Koszul and homogeneous Koszul algebras were introduced by Priddy in [18]; despite of that these type of algebras have not been enough studied, they have important applications in algebraic geometry, Lie theory, quantum groups, algebraic topology and combinatorics. The structure and history of Koszul homogeneous algebras were detailed in [17]. There exist numerous equivalent definitions of homogeneous Koszul algebras (see for example [3]); in addition, Koszul algebras have been defined in a more general way by some authors and they are commonly called “Generalized Koszul algebras” (see for example [4], [7], [14], [27]). In this paper we work with the definition given by Priddy, which is commonly called “Koszul algebras in the classical sense”.

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Skew PBW extensions or σ -PBW extensions were defined in [8]. Several properties of these extensions have been recently studied (see for example [1], [2], [9], [10], [11], [12], [19], [20], [21], [22], [25], [26]). There exist some relations between Koszul and homogeneous Koszul algebras with the skew PBW extensions. Our interest in this article is to study the Koszul property for the skew PBW extensions over fields. For this purpose we classify the skew PBW extensions in five sub-classes: constant, bijective, pre-commutative, quasi-commutative and semi-commutative, and we show that a skew PBW extension A of a field is Koszul (homogeneous Koszul) when A is pre-commutative and constant (semi-commutative). Finally, following the ideas presented in [6], we show that a constant skew PBW extension of a field is a PBW deformation of its homogeneous version.

2 Skew PBW extensions

In this section we recall some elementary properties of skew PBW extensions; in addition, we will introduce some sub-classes of them: constant, pre-commutative and semi-commutative. Examples of these sub-classes are presented.

2.1 Definitions and properties

Definition 2.1 ([8], Definition 1). Let R and A be rings. We say that A is a *skew PBW extension of R* (also called a σ -PBW extension of R) if the following conditions hold:

- (i) $R \subseteq A$;
- (ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such A is a left R -free module with basis

$$\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}, \text{ with } \mathbb{N} := \{0, 1, 2, \dots\}.$$

The set $\text{Mon}(A)$ is called the set of standard monomials of A .

- (iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{2.1}$$

- (iv) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2.2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n \rangle$.

The notation $\sigma(R)\langle x_1, \dots, x_n \rangle$ and the name of the skew PBW extensions are due to the following proposition.

Proposition 2.2 ([8], Proposition 3). *Let A be a skew PBW extension of R . For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \rightarrow R$ and a σ_i -derivation $\delta_i : R \rightarrow R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \quad r \in R. \quad (2.3)$$

Definition 2.3. Let A be a skew PBW extension of R , $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$, where σ_i and δ_i ($1 \leq i \leq n$) are as in the Proposition 2.2.

- (a) A is called *pre-commutative* if the condition (iv) in Definition 2.1 is replaced by:
For any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in Rx_1 + \dots + Rx_n. \quad (2.4)$$

- (b) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by

- (iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i; \quad (2.5)$$

- (iv') for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \quad (2.6)$$

- (c) A is called *bijective* if σ_i is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.
- (d) Any element r of R such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$ will be called a *constant*. A is called *constant* if every element of R is constant.
- (e) A is called *semi-commutative* if A is quasi-commutative and constant.

Recall that a *filtered ring* is a ring B with a family $F(B) = \{F_n(B) \mid n \in \mathbb{Z}\}$ of subgroups of the additive group of B where we have the ascending chain $\dots \subset F_{n-1}(B) \subset F_n(B) \subset \dots$ such that $1 \in F_0(B)$ and $F_n(B)F_m(B) \subseteq F_{n+m}(B)$ for all $n, m \in \mathbb{Z}$. From a filtered ring B it is possible to construct its associated graded ring $Gr(B)$ taking $Gr(B)_n := F_n(B)/F_{n-1}(B)$. The following proposition establishes that one can construct a quasi-commutative skew PBW extension from a given skew PBW extension of a ring R .

Proposition 2.4 ([12], Proposition 2.1). *Let A be a skew PBW extension of R . Then, there exists a quasicommutative skew PBW extension A^σ of R in n variables z_1, \dots, z_n defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$, where $c_{i,r}, c_{i,j}$ are the same constants that define A . Moreover, if A is bijective then A^σ is also bijective.*

The next proposition computes the graduation of a skew PBW extension.

Theorem 2.5 ([12], Theorem 2.2). *Let A be an arbitrary skew PBW extension of R . Then, A is a filtered ring with increasing filtration given by*

$$F_m(A) := \begin{cases} R & \text{if } m = 0 \\ \{f \in A \mid \deg(f) \leq m\} & \text{if } m \geq 1 \end{cases} \quad (2.7)$$

and the corresponding graded ring $Gr(A)$ is isomorphic to A^σ .

2.2 Examples and classification

Examples 2.6. In [8] and [12] was presented a list of examples of quasi-commutative or bijective skew PBW extensions. We also classify these examples according to Definition 2.3. Through this paper, \mathbb{K} will denote a field and K a commutative ring.

1. Classical polynomial ring; Ore extensions of bijective type and Weyl algebras; Universal enveloping algebra of a Lie algebra; Tensor product; crossed product; Algebra of q -differential operators; Algebra of shift operators; Mixed algebras; Algebra of discrete linear systems; Linear partial differential operators; Linear partial shift operators; Algebra of linear partial difference operators; Algebra of linear partial q -dilation operators; Algebra of linear partial q -differential operators; Diffusion algebra 1 ([21]); Diffusion algebra 2 ([12]); Additive analogue of the Weyl algebra; Multiplicative analogue of the Weyl algebras; Quantum algebras; Dispin algebras; Woronowicz algebras; Complex algebras; Algebra \mathbf{U} ; Manin algebras; Algebra of quantum matrices; q -Heisenberg algebras; Quantum enveloping algebras of $\mathfrak{sl}(2, \mathbb{K})$; The algebra of differential operators on a quantum space; Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2, \mathbb{K}))$; Quantum Weyl algebra of Mal'sinotis; Quantum Weyl algebras; Multiparameter quantized Weyl algebras; Quantum symplectic space and Quadratic algebras in 3 variables.
2. Jordan plane. The Jordan plane A is the free \mathbb{K} -algebra generated by x, y with relation $yx = xy + x^2$, so $A = \mathbb{K}\langle x, y \rangle / \langle yx - xy - x^2 \rangle \cong \sigma(\mathbb{K}[x])\langle y \rangle$.
3. Particular Sklyanin algebra. The Sklyanin algebra (Example 1.14, [23]) is the \mathbb{K} -algebra $S = \mathbb{K}\langle x, y, z \rangle / \langle ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cx^2 \rangle$, where $a, b, c \in \mathbb{K}$. If $c \neq 0$ then S is not a skew PBW extension. If $c = 0$ and $a, b \neq 0$ then in S : $yx = -\frac{b}{a}xy$; $zx = -\frac{a}{b}xz$ and $zy = -\frac{b}{a}yz$, therefore $S \cong \sigma(\mathbb{K})\langle x, y, z \rangle$ is a skew PBW extension of \mathbb{K} , and we call this algebra a *particular Sklyanin algebra*.
4. Multi-parameter quantum affine n -spaces. Let $n \geq 1$ and \mathbf{q} be a matrix $(q_{ij})_{n \times n}$ with entries in a field \mathbb{K} where $q_{ii} = 1$ y $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. Then multi-parameter quantum affine n -space $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ is defined to be \mathbb{K} -algebra generated by x_1, \dots, x_n with the relations $x_j x_i = q_{ij} x_i x_j$ for all $1 \leq i, j \leq n$.

5. Homogenized enveloping algebra ([24], Chapter 12). Let \mathcal{G} a finite dimensional Lie algebra over \mathbb{K} with basis $\{x_1, \dots, x_n\}$ and $\mathcal{U}(\mathcal{G})$ its enveloping algebra. The *homogenized enveloping algebra* of \mathcal{G} is $\mathcal{A}(\mathcal{G}) := T(\mathcal{G} \oplus \mathbb{K}z)/\langle R \rangle$, where $T(\mathcal{G} \oplus \mathbb{K}z)$ is the tensor algebra, z is a new variable, and R is spanned by $\{z \otimes x - x \otimes z \mid x \in \mathcal{G}\} \cup \{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in \mathcal{G}\}$. From the PBW Theorem for $\mathcal{G} \otimes \mathbb{K}(z)$, considered as a Lie algebra over $\mathbb{K}(z)$, we get that $\mathcal{A}(\mathcal{G})$ is a skew PBW extension of $\mathbb{K}[z]$.

Classification 2.7. We classify the above examples of skew PBW extensions as constant (C), bijective (B), pre-commutative (P), quasi-commutative (QC) and semi-commutative (SC); the classification is presented in the next table, where the symbols \star and \checkmark denote negation and affirmation, respectively.

Classification of Examples 2.6					
Skew PBW extension	C	B	P	QC	SC
Classical polynomial ring	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
Ore extensions of bijective type	\star	\checkmark	\checkmark	\star	\star
Weyl algebra	\star	\checkmark	\checkmark	\star	\star
Jordan plane	\star	\checkmark	\checkmark	\star	\star
Particular Sklyanin algebra	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
Universal enveloping algebra of a Lie algebra	\checkmark	\checkmark	\checkmark	\star	\star
Homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$	\checkmark	\checkmark	\checkmark	\star	\star
Tensor product	\checkmark	\checkmark	\checkmark	\star	\star
Crossed product	\star	\checkmark	\star	\star	\star
Algebra of q -differential operators	\star	\checkmark	\checkmark	\star	\star
Algebra of shift operators	\star	\checkmark	\checkmark	\checkmark	\star
Mixed algebra	\star	\checkmark	\star	\star	\star
Algebra of discrete linear systems	\star	\checkmark	\checkmark	\checkmark	\star
Linear partial differential operators	\star	\checkmark	\checkmark	\star	\star
Linear partial shift operators	\star	\checkmark	\checkmark	\checkmark	\star
Algebra of linear partial difference operators	\star	\checkmark	\checkmark	\star	\star
Algebra of linear partial q -dilation operators	\star	\checkmark	\checkmark	\checkmark	\checkmark
Algebra of linear partial q -differential operators	\star	\checkmark	\checkmark	\star	\star
Diffusion algebra 1	\checkmark	\checkmark	\checkmark	\star	\star
Diffusion algebra 2	\checkmark	\checkmark	\checkmark	\star	\star
Additive analogue of the Weyl algebra	\checkmark	\checkmark	\star	\star	\star
Multiplicative analogue of the Weyl algebra	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
Quantum algebra	\checkmark	\checkmark	\checkmark	\star	\star
Dispin algebra	\checkmark	\checkmark	\checkmark	\star	\star
Woronowicz algebra	\checkmark	\checkmark	\checkmark	\star	\star
Complex algebra	\star	\checkmark	\star	\star	\star
Algebra \mathbf{U}	\star	\checkmark	\star	\star	\star
Manin algebra	\star	\checkmark	\checkmark	\star	\star
q -Heisenberg algebra	\checkmark	\checkmark	\checkmark	\star	\star
Quantum enveloping algebra of $\mathfrak{sl}(2, \mathbb{K})$	\star	\checkmark	\star	\star	\star
Hayashi's algebra	\star	\checkmark	\star	\star	\star
Multi-parameter quantum affine n -space	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
The algebra of differential operators on a quantum space S_q	\star	\checkmark	\star	\star	\star
Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2, \mathbb{K}))$	\star	\checkmark	\star	\star	\star
Quantum Weyl algebra of Maltiniotis	\star	\checkmark	\star	\star	\star
Quantum Weyl algebra	\star	\checkmark	\star	\star	\star
Multiparameter quantized Weyl algebra	\star	\checkmark	\star	\star	\star
Quantum symplectic space	\star	\checkmark	\star	\star	\star
Quadratic algebras in 3 variable	\star	\checkmark	\star	\star	\star

Example 2.8. *Sridharan enveloping algebra of 3-dimensional Lie algebra \mathcal{G} .* Let \mathcal{G} be a finite dimensional Lie algebra, and let $f \in Z^2(\mathcal{G}, \mathbb{K})$ be an arbitrary 2-cocycle, that is, $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{K}$ such that $f(x, x) = 0$ and

$$f(x, [y, z]) + f(z, [x, y]) + f(y, [z, x]) = 0$$

for all $x, y, z \in \mathcal{G}$. The Sridharan enveloping algebra of \mathcal{G} is defined to be the associative algebra $\mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I$, where $T(\mathcal{G})$ is the tensor algebra of \mathcal{G} and I is the two-sided ideal of $T(\mathcal{G})$ generated by the elements

$$(x \otimes y) - (y \otimes x) - [x, y] - f(x, y), \text{ for all } x, y \in \mathcal{G}.$$

Note that if $f = 0$ then $\mathcal{U}_f(\mathcal{G}) = \mathcal{U}_0(\mathcal{G}) = \mathcal{U}(\mathcal{G})$. For $x \in \mathcal{G}$, we still denote by x its image in $\mathcal{U}_f(\mathcal{G})$. $\mathcal{U}_f(\mathcal{G})$ is a filtered algebra with the associated graded algebra $gr(\mathcal{U}_f(\mathcal{G}))$ being a polynomial algebra. Let \mathbb{K} be an algebraically closed field of characteristic zero. If \mathcal{G} is a Lie \mathbb{K} -algebra of dimension three, then the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ for $f \in Z^2(\mathcal{G}, \mathbb{K})$ is isomorphic to one of ten associative \mathbb{K} -algebras (see [16], Theorem 1.3), which is defined by three generators x, y, z and the commutation relations as the following table shows. Therefore, the Sridharan enveloping algebra $\mathcal{U}_f(\mathcal{G})$ is a skew PBW extension of \mathbb{K} , i.e. $\mathcal{U}_f(\mathcal{G}) \cong \sigma(\mathbb{K})\langle x, y, z \rangle$, and it is classified as follows:

Sridharan enveloping algebra of 3-dimensional Lie algebra \mathcal{G}								
Type	$[x, y]$	$[y, z]$	$[z, x]$	C	B	P	QC	SC
1	0	0	0	✓	✓	✓	✓	✓
2	0	x	0	✓	✓	✓	*	*
3	x	0	0	✓	✓	✓	*	*
4	0	αy	$-x$	✓	✓	✓	*	*
5	0	y	$-(x + y)$	✓	✓	✓	*	*
6	z	$-2y$	$-2x$	✓	✓	✓	*	*
7	1	0	0	✓	✓	*	*	*
8	1	x	0	✓	✓	*	*	*
9	x	1	0	✓	✓	*	*	*
10	1	y	x	✓	✓	*	*	*

where $\alpha \in \mathbb{K} \setminus \{0\}$.

3 Koszulity

Some authors have defined Koszul algebras in a more general sense than [18] (see for example [4], [14], [27]). Our focus is to study the Koszul property for skew PBW extensions taking into account the definition given in [18]. In this section we give sufficient conditions to guarantee that skew PBW extensions are Koszul or homogeneous Koszul. For this purpose, we show the relationship between \mathbb{K} -algebras that are skew PBW extensions and certain classes of algebras defined in [18] containing the Koszul and homogeneous Koszul algebras. Let $L := \mathbb{K}\langle x_1, \dots, x_n \rangle$ the free associative algebra (tensor algebra) in n generators x_1, \dots, x_n . Note that L is positively graded with graduation given by $L := \bigoplus_{j \geq 0} L_j$, where $L_0 = \mathbb{K}$ and L_j spanned by all words of length j in the alphabet $\{x_1, \dots, x_n\}$, for $j > 0$.

3.1 Pre-Koszul algebras

We present a definition of pre-Koszul and homogeneous pre-Koszul algebras, analogous to the definition given by Priddy in [18].

Definition 3.1. Let $L = \mathbb{K}\langle x_1, \dots, x_n \rangle$ and let $B := L/I$.

- (i) B is said to be a *pre-Koszul algebra* if I is a two sided ideal generated by elements of the form

$$\sum_{i=1}^n c_i x_i + \sum_{1 \leq j, k \leq n} c_{j,k} x_j x_k, \text{ where } c_i \text{ and } c_{j,k} \text{ are in } \mathbb{K}, \quad (3.1)$$

- (ii) A pre-Koszul algebra is said to be *pre-Koszul homogeneous* if $c_i = 0$, for $1 \leq i \leq n$ in (3.1).

Presentations of special types of skew PBW extensions are given in the following remark.

Remark 3.2. Let $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ be a skew PBW extension of a field \mathbb{K} .

1. We note that $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two sided ideal generated by elements as in (iii) and (iv) of the Definition 2.1, i.e., elements of the form

$$c_r + x_i r - c_{i,r} x_i, \quad r_0 + r_1 x_1 + \dots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \quad (3.2)$$

where $r \neq 0$, $c_r, c_{i,r} \neq 0$, $r_0, r_1, \dots, r_n, c_{i,j} \neq 0$ are elements in \mathbb{K} , with $1 \leq i, j \leq n$.

2. If A is pre-commutative, then $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two sided ideal generated by elements of the form

$$c_r + x_i r - c_{i,r} x_i, \quad r_1 x_1 + \dots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \quad (3.3)$$

where $r \neq 0$, $c_r, c_{i,r} \neq 0$, $r_1, \dots, r_n, c_{i,j} \neq 0$ are elements in \mathbb{K} , with $1 \leq i, j \leq n$.

3. If A is constant, then $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two sided ideal generated by elements of the form

$$r_0 + r_1 x_1 + \dots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \quad (3.4)$$

where $r_0, r_1, \dots, r_n, c_{i,j} \neq 0$ are elements in \mathbb{K} , with $1 \leq i, j \leq n$.

4. If A is quasi-commutative then $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two sided ideal generated by elements as in (iii') and (iv') of the Definition 2.3, i.e., elements of the form

$$x_i r - c_{i,r} x_i, \quad x_j x_i - c_{i,j} x_i x_j \quad (3.5)$$

where $r \neq 0$, $c_{i,r} \neq 0$, $c_{i,j} \neq 0$ are elements in \mathbb{K} , with $1 \leq i, j \leq n$.

5. If A is semi-commutative then $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two sided ideal generated by elements of the form

$$x_j x_i - c_{i,j} x_i x_j \quad (3.6)$$

where $c_{i,j} \neq 0$ are elements in \mathbb{K} , with $1 \leq i, j \leq n$.

If otherwise is not assumed, in this paper all skew PBW extensions are \mathbb{K} -algebras and extensions of the field \mathbb{K} (i.e., $R = \mathbb{K}$ in Definition 2.1), so $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ is necessarily a constant skew PBW extension.

Proposition 3.3. *Let $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ be a skew PBW extension. If A is pre-commutative then A is pre-Koszul.*

Proof. From (3.3) and (3.4) we have that $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two sided ideal generated by elements of the form

$$r_1 x_1 + \dots + r_n x_n + x_j x_i - c_{i,j} x_i x_j. \quad (3.7)$$

Then we conclude that A is pre-Koszul. \square

Example 3.4. According to the classifications presented in the tables of Section 2, the following skew PBW extensions are pre-Koszul algebras: classical polynomial ring over a field; particular Sklyanin algebra; universal enveloping algebra of a Lie algebra; algebra of linear partial q -dilation operators; additive analogue of the Weyl algebra; multiplicative analogue of the Weyl algebra; quantum algebra $\mathcal{U}'(so(3, K))$; dispin algebra; Woronowicz algebra; q -Heisenberg algebra; multi-parameter quantum affine n -space; types 1, 2, 3, 4, 5 and 6 of Sridharan enveloping algebra of 3-dimensional Lie algebras.

Proposition 3.5. *Let A be a skew PBW extension. If A is semi-commutative then A is pre-Koszul homogeneous.*

Proof. If A is a semi-commutative skew PBW extension then A from (3.6) and Proposition 3.3 we get that A is pre-Koszul homogeneous. \square

Example 3.6. From Example 3.4 we obtain the following examples of pre-Koszul homogeneous skew PBW extensions: classical polynomial ring over a field; particular Sklyanin algebras; multiplicative analogue of the Weyl algebra; multi-parameter quantum affine n -space; the Sridharan enveloping algebra of 3-dimensional Lie algebra with $[x, y] = [y, z] = [z, x] = 0$.

Let B be a pre-Koszul algebra. One can truncate the relations in (3.1) leaving only their homogeneous quadratic parts. Let $B^{(0)}$ be the obtained algebra. Then $B^{(0)}$ is called the *associated homogeneous pre-Koszul algebra* of B . Note that B is homogeneous if and only if $B^{(0)} \cong B$ as algebras.

Proposition 3.7. *Let A be a pre-Koszul skew PBW extension, then A^σ is the associated homogeneous pre-Koszul algebra of A .*

Proof. Let A be a pre-Koszul skew PBW extension. By Proposition 2.4 there exists a quasi-commutative skew PBW extension A^σ of \mathbb{K} in n variables z_1, \dots, z_n defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$, where $c_{i,r}, c_{i,j}$ are the same constants that define A . Since A is pre-Koszul then by Proposition 3.3 A is constant and therefore A^σ is defined by the relations $z_j z_i = c_{i,j} z_i z_j$. Then $A^{(0)} \cong A^\sigma$. \square

3.2 Koszul algebras and skew PBW extensions

Let B be a finitely graded algebra generated in degree 1; consider the *Yoneda algebra* of B defined by

$$E(B) := \bigoplus_{i \geq 0} \text{Ext}_B^i(\mathbb{K}, \mathbb{K});$$

the Ext groups here are computed in the category of graded B -modules with graded Hom spaces; the product in $E(B)$ is defined in the following way: Let $\{P_i \xrightarrow{d_i} P_{i-1}\}_{i \geq 0}$ be a graded projective resolution of \mathbb{K} that defines the groups $\text{Ext}_B^i(\mathbb{K}, \mathbb{K})$, with $P_{-1} := \mathbb{K}$; moreover, let $\bar{f} \in \text{Ext}_B^i(\mathbb{K}, \mathbb{K}) = \ker(d_{i+1}^*) / \text{Im} f_i^*$ with $f \in \ker(d_{i+1}^*) \subseteq \text{Hom}_B(P_i, \mathbb{K})$ and $\bar{g} \in \text{Ext}_B^j(\mathbb{K}, \mathbb{K}) = \ker(d_{j+1}^*) / \text{Im} f_j^*$ with $g \in \ker(d_{j+1}^*) \subseteq \text{Hom}_B(P_j, \mathbb{K})$, then we define

$$\begin{aligned} \text{Ext}_B^i(\mathbb{K}, \mathbb{K}) \times \text{Ext}_B^j(\mathbb{K}, \mathbb{K}) &\rightarrow \text{Ext}_B^{i+j}(\mathbb{K}, \mathbb{K}) \\ (\bar{f}, \bar{g}) &\mapsto \bar{f} \bar{g} := \overline{f g'}, \end{aligned}$$

where $g' : P_{i+j} \rightarrow P_i$ is defined inductively by the following commutative diagrams:

$$\begin{array}{ccccc} & & P_j & & P_{j+1} & & P_{j+i} \\ & \nearrow g_0 & \downarrow g & \Rightarrow & \nearrow g_1 & \downarrow g_0 d_{j+1} & \Rightarrow \dots \Rightarrow & \nearrow g' := g_i & \downarrow g_{i-1} d_{j+i} \\ P_0 & \xrightarrow{d_0} & \mathbb{K} & & P_1 & \xrightarrow{d_1} & \text{Im}(d_1) & & P_i & \xrightarrow{d_i} & \text{Im}(d_i) \end{array}$$

Can be proved that this product is well defined, i.e., it does not depend of the projective resolution of \mathbb{K} and the choosing of $g_0, g_1, \dots, g_{i-1}, g_i$; moreover, $f g' \in \ker(d_{i+j+1}^*)$: In fact, from the step $i+1$ in the previous inductive procedure we have that $d_{i+1} g_{i+1} = g_i d_{i+j+1}$, so $f d_{i+1} g_{i+1} = f g_i d_{i+j+1}$, i.e., $0 = d_{i+1}^*(f) g_{i+1} = d_{i+j+1}^*(f g_i)$.

Thus, $E(B)$ is a graded algebra; note that the \mathbb{K} -vector space $\text{Ext}_B^i(\mathbb{K}, \mathbb{K})$ is graded

$$\text{Ext}_B^i(\mathbb{K}, \mathbb{K}) = \bigoplus_{j \geq 0} \text{Ext}_B^{i,j}(\mathbb{K}, \mathbb{K}),$$

with

$$Ext_B^{i,j}(\mathbb{K}, \mathbb{K}) := (Ext_B^i(\mathbb{K}, \mathbb{K}))_{-j} := Ext_B^i(\mathbb{K}, \mathbb{K}(-j)),$$

so setting $E^{i,j}(B) := Ext_B^{i,j}(\mathbb{K}, \mathbb{K})$ we get that

$$E(B) = \bigoplus_{i,j \geq 0} E^{i,j}(B)$$

is a bigraded algebra. For $i \geq 0$, we write

$$E^i(B) := \bigoplus_{j \geq 0} E^{i,j}(B);$$

in particular,

$$E^0(B) = \bigoplus_{j \geq 0} Hom_B^j(\mathbb{K}, \mathbb{K}) = \bigoplus_{j \geq 0} (Hom_B(\mathbb{K}, \mathbb{K}))_{-j} = \bigoplus_{j \geq 0} Hom_B(\mathbb{K}, \mathbb{K}(-j)),$$

with $Hom_B(\mathbb{K}, \mathbb{K}(-j)) := \{f \in Hom_B(\mathbb{K}, \mathbb{K}) | f(\mathbb{K}_l) \subseteq \mathbb{K}_{l-j}, l \in \mathbb{Z}\}$.

Definition 3.8. Let B be a homogeneous pre-Koszul algebra, B is called *homogeneous Koszul* if the following equivalent conditions hold:

- (i) $Ext_B^{i,j}(\mathbb{K}, \mathbb{K}) = 0$ for $i \neq j$;
- (ii) $E(B)$ is generated by $E^{1,1}(B)$;
- (iii) The module \mathbb{K} admits a *linear free resolution*, i.e., a resolution by free B -modules

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{K} \rightarrow 0$$

such that P_i is generated in degree i .

Definition 3.9 ([18], Page 43). We say that a pre-Koszul algebra B is a *Koszul algebra* if $B^{(0)}$ is a homogeneous Koszul algebra.

Remark 3.10. Notice that if B is homogeneous Koszul algebra then B is Koszul. In fact, as B is homogeneous then $B^{(0)} \cong B$ as algebras and so $B^{(0)}$ is homogeneous Koszul, therefore B is Koszul.

In the current literature, homogeneous Koszul algebras are called simply Koszul algebras. Some authors have studied Koszul algebras defined by Priddy in [18]. For example Koszul algebras are defined in [17], analogous to the Definition 3.9. Let $P \subseteq \mathbb{K} \oplus L_1 \oplus L_2$ a subspace of $F_2(L)$ and $A = L/\langle P \rangle$. Let $A^{(0)} = L/\langle R \rangle$, where R is obtained by taking homogeneous part of P . A is said to be (nonhomogeneous) Koszul if $A^{(0)}$ is homogeneous Koszul (see [17], page 140). In [15] Koszul algebras are defined as follows. Let V a graded vector space and a degree homogeneous subspace $P \subseteq V \oplus V^{\otimes 2}$, the algebra $A = T(V)/\langle P \rangle$ is called (nonhomogeneous quadratic) Koszul if $P \cap V = \{0\}$, $\{P \otimes V + V \otimes P\} \cap V^{\otimes 2} \subseteq P \cap V^{\otimes 2}$ and $T(V)/\langle \pi(P) \rangle$ is homogeneous Koszul, where $\pi : T(V) \rightarrow V^{\otimes 2}$ is the projection onto the quadratic part of the tensor algebra. R. Berger in [5] defined the notion of N -Koszul algebra, if $N = 2$, the notion of homogeneous Koszul

algebra is obtained. To avoid confusion, we still use the names given in the Definition 3.8 (homogeneous Koszul) and the Definition 3.9 (Koszul).

Let $L = \mathbb{K}\langle x_1, \dots, x_n \rangle$ the free associative algebra in n generators x_1, \dots, x_n . Let R a subspace of $F_2(L) = \mathbb{K} \oplus L_1 \oplus L_2$, the algebra $L/\langle R \rangle$ is called (nonhomogeneous) *quadratic algebra*. $L/\langle R \rangle$ is called *homogeneous quadratic algebra* if R is a subspace of L_2 , for $\langle R \rangle$ the two-sided ideal of L generated by R . Let $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle R \rangle$ be a quadratic algebra with a fixed generators $\{x_1, \dots, x_n\}$. For a multiindex $\alpha := (i_1, \dots, i_m)$, where $1 \leq i_k \leq n$, we denote the monomials in $\mathbb{K}\langle x_1, \dots, x_n \rangle$ by $x^\alpha := x_{i_1}x_{i_2} \cdots x_{i_m}$. For $\alpha = \emptyset$ we set $x^\emptyset := 1$. Now let us equip the subspace L_2 with the basis consisting of the monomials $x_{i_1}x_{i_2}$. Let $S^{(1)} := \{1, 2, \dots, n\}$, $S^{(1)} \times S^{(1)}$ the cartesian product, then for $R \subseteq L_2$ we obtain the set $S \subseteq S_1 \times S_1$ of pairs of indices (l, m) for which the class of x_lx_m in L_2/R is not in the span of the classes of x_rx_s with $(r, s) < (l, m)$, where $<$ denotes the lexicographical order ([17], 4.1-Lemma 1.1). Hence, the relations in A can be written in the following form:

$$x_ix_j = \sum_{\substack{(r,s) < (i,j) \\ (r,s) \in S}} c_{ij}^{rs} x_rx_s, \quad (i, j) \in S^{(1)} \times S^{(1)} \setminus S.$$

Define further $S^{(0)} := \{\emptyset\}$, and for $m \geq 2$,

$$S^{(m)} := \{(i_1, \dots, i_m) \mid (i_k, i_{k+1}) \in S, k = 1, \dots, m-1\}$$

and consider the monomials $\{x_{i_1} \cdots x_{i_m} \in A_m \mid (i_1, \dots, i_m) \in S^{(m)}\}$. Note that these monomials always span A_m as a vector space and the monomials

$$(A, S) := \{x_{i_1} \cdots x_{i_m} \mid (i_1, \dots, i_m) \in \cup_{m \geq 0} S^{(m)}\} \quad (3.8)$$

linearly span the entire A . We call (A, S) in (3.8) a *PBW-basis* of A if they are linearly independent and hence form a \mathbb{K} -linear basis. The elements x_1, \dots, x_n are called *PBW-generators* of A . A *PBW-algebra* is a homogeneous quadratic algebra admitting a PBW-basis, i.e., there exists a permutation of x_1, \dots, x_n such that the standard monomials in x_1, \dots, x_n conform a \mathbb{K} -basis of A .

Proposition 3.11. *Let A be a semi-commutative skew PBW extension. Then A is a PBW algebra.*

Proof. If $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ is a semi-commutative skew PBW extension, then $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle x_jx_i - c_{i,j}x_ix_j \rangle$ (as in (3.6)) is a homogeneous quadratic algebra with generators x_1, \dots, x_n and relations $x_jx_i - c_{i,j}x_ix_j$. Using the above notation we have that for $1 \leq i \leq j \leq n$, the class of x_ix_j is not in the span of the classes of x_rx_s with $(r, s) < (i, j)$, but, the class of x_jx_i is in the span of the class

of $x_i x_j$ with $(i, j) < (j, i)$. Therefore $S = \{(i, j) \mid 1 \leq i \leq j \leq n\} = S^{(2)}$ and $S^{(m)} = \{(i_1, \dots, i_m) \mid i_1 \leq i_2 \leq \dots \leq i_m, 1 \leq i_k \leq n\}$ for $m \geq 3$. Then

$$(A, S) = \{x_1^{m_1} \cdots x_n^{m_n} \mid m_1, \dots, m_n \geq 0\} = \text{Mon}(A) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

By Definition 2.1 (ii), $\text{Mon}(A)$ is a \mathbb{K} -basis for A and therefore A is a PBW algebra. \square

Theorem 3.12 ([18], Theorem 5.3). *If B is a PBW algebra then B is a homogeneous Koszul algebra.*

The Theorem 3.12 and this proof can also be found in [17], Theorem 3.1, page 84; there they also present an example of a homogeneous Koszul algebra which is not PBW algebra.

Corollary 3.13. *Every semi-commutative skew PBW extension is homogeneous Koszul algebra.*

Proof. If following from Proposition 3.11 and Theorem 3.12. \square

Theorem 3.14. *Every pre-commutative skew PBW extension is Koszul.*

Proof. If A is a pre-commutative skew PBW extension then by Remark 3.2, $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$, where I is the two-sided ideal generated by relations of the form $x_j x_i - c_{i,j} x_i x_j + \sum_{t=1}^n k_t x_t$, $0 \neq c_{i,j}, k_t \in \mathbb{K}$, $1 \leq i, j, t \leq n$. By Proposition 3.3 A is pre-Koszul, therefore from Proposition 3.7, $A^{(0)} = A^\sigma = \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle x_j x_i - c_{i,j} x_i x_j \rangle$ is the associated homogeneous pre-Koszul algebra of A . Note that A^σ is semi-commutative, so by Corollary 3.13, $A^{(0)}$ is a homogeneous Koszul algebra, i.e., A is Koszul. \square

Corollary 3.15. *If A is a pre-commutative skew PBW extension then $\text{Gr}(A)$ is homogeneous Koszul.*

Examples 3.16. Next we present some examples of homogeneous Koszul skew PBW extensions, many of which had already been presented by other authors with the name of Koszul algebras. For this purpose we use the classification given in Subsection 2.2 and Corollary 3.13: classical polynomial ring; particular Sklyanin algebra; Algebra of linear partial q -dilation operators; multiplicative analogue of the Weyl algebra; multi-parameter quantum affine n -spaces; the Sridharan enveloping algebra of 3-dimensional Lie algebra with $[x, y] = [y, z] = [z, x] = 0$.

Examples 3.17. Recall that every homogeneous Koszul algebra is Koszul (Remark 3.10), so Examples 3.16 are Koszul skew PBW extensions. According to classification given in Subsection 2.2 and Theorem 3.14 the next skew PBW extensions are Koszul: universal enveloping algebra of a Lie algebra, with \mathbb{K} a field; diffusion algebra 1; quantum algebra; Dispian algebra; Woronowicz algebra; q -Heisenberg algebra; types 1, 2, 3, 4, 5 and 6 of Sridharan enveloping algebra of 3-dimensional Lie algebra (Example 2.8).

Note that some particular classes of skew PBW extensions in Examples 3.16 and 3.17 represent the same algebra. For example, Sridharan enveloping algebra of 3-dimensional Lie algebra of type 1 and the classical polynomial ring $\mathbb{K}[x, y, z]$ are the same algebra.

4 PBW deformations

Let V be a vector space over a field \mathbb{K} and let $T(V) = \bigoplus T^i(V)$ be its tensor algebra over \mathbb{K} . Consider the natural filtration $F_i(T) = \{\bigoplus T^j(V) \mid j \leq i\}$ of $T(V)$. Fix a subspace $P \subseteq F_2(T) = \mathbb{K} \oplus V \oplus (V \otimes V)$, and let us consider the two-sided ideal $\langle P \rangle$ in $T(V)$ generated by P . Let $A = T(V)/\langle P \rangle$ be a nonhomogeneous quadratic algebra. It inherits a filtration $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ from $T(V)$, let $Gr(A)$ the associated graded algebra. Consider the natural projection $\pi : F_2(T) = \mathbb{K} \oplus V \oplus (V \otimes V) \rightarrow V \otimes V$ on the homogeneous component, set $R = \pi(P)$ and consider the homogeneous quadratic algebra $T(V)/\langle R \rangle$. $T(V)/\langle R \rangle$ is called the *homogeneous version* (or the *induced homogeneous quadratic*) algebra of A determined by P . We have the natural epimorphism $p : T(V)/\langle R \rangle \rightarrow Gr(A)$ (induced by the projection $T(V) \rightarrow A$).

Definition 4.1 ([6], Page 316). With the above notation, a nonhomogeneous quadratic algebra $A := T(V)/\langle P \rangle$ is a *Poincaré-Birkhoff-Witt (PBW) deformation* (or satisfies the PBW *property* with respect to the subspace P of $F_2(T)$) of $B := T(V)/\langle R \rangle$ if the natural projection $p : T(V)/\langle R \rangle \rightarrow Gr(A)$ is an isomorphism.

Proposition 4.2 ([15], Theorem 3.6.4). *Let $A = T(V)/\langle P \rangle$ with $P \subseteq V \oplus V^{\otimes 2}$, $P \cap V = \{0\}$ and $\{P \otimes V + V \otimes P\} \cap V^{\otimes 2} \subseteq P \cap V^{\otimes 2}$. If A is Koszul, then the epimorphism $p : B = T(V)/\langle R \rangle \rightarrow Gr(A)$ is an isomorphism of graded algebras, i.e., A is a PBW deformation of B .*

Corollary 4.3 ([15], Corollary 3.6.5). *Let $A = T(V)/\langle P \rangle$, with $P \subseteq V \oplus V^{\otimes 2}$. If $B = T(V)/\langle R \rangle$ is homogeneous Koszul, then:*

1. $P \cap V = \{0\} \Leftrightarrow \langle P \rangle \cap V = \{0\}$
2. $\{P \otimes V + V \otimes P\} \cap V^{\otimes 2} \subseteq P \cap V^{\otimes 2} \Leftrightarrow \langle P \rangle \cap \{V \oplus V^{\otimes 2}\} = P$.

Remark 4.4. [6], Lemma 0.4 establishes that if the algebra $T(V)/\langle P \rangle$ is a PBW deformation of $T(V)/\langle R \rangle$ then it satisfies the following conditions:

- (I) $P \cap F_1(T) = 0$;
- (J) $(F_1(T) \cdot P \cdot F_1(T)) \cap F_2(T) = P$.

If a nonhomogeneous quadratic algebras satisfy (I) then the subspace $P \subset F_2(T)$ can be described in terms of two maps $\alpha : R \rightarrow V$ and $\beta : R \rightarrow \mathbb{K}$ as $P = \{x - \alpha(x) - \beta(x) \mid x \in R\}$. If $A = T(V)/\langle P \rangle$ is a PBW deformations of its homogeneous version then P can not have relations in $F_1(T)$, so:

- (i) If A is a PBW deformation of some skew PBW extension B , then A is constant.

- (ii) The homogeneous version of a skew PBW extension A is the skew PBW extension B such that the conditions (i) and (ii) of Definition 2.1 for A are satisfied for B , and the conditions (iii) and (iv) are replaced by $x_j x_i - c_{i,j} x_i x_j = 0$, where $c_{i,j}$ are the same as for A .
- (iii) The homogeneous version of a skew PBW extension is homogeneous Koszul.

For example the homogeneous version for the universal enveloping algebra of a Lie algebra \mathcal{G} , $\mathcal{U}(\mathcal{G})$ is the symmetric algebra $\mathbb{S}(\mathcal{G})$.

Proposition 4.5. *Let A be a constant skew PBW extension of a field \mathbb{K} . Then A is a PBW deformation of its homogeneous version B .*

Proof. Let A be a constant skew PBW extension of a field \mathbb{K} then, $x_j x_i - c_{i,j} x_i x_j + r_0 + r_1 x_1 + \dots + r_n x_n$ (as in Definition 2.1) are the generated relations of the subspace P , that is, $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle P \rangle$. Then the subspace $\pi(P) = R$ is generated by the relations $x_j x_i - c_{i,j} x_i x_j$, i.e., $\mathbb{K}\langle x_1, \dots, x_n \rangle / \langle R \rangle = B$ is the homogeneous version of A . Now for Theorem 2.5, $Gr(A) \cong A^\sigma$ where A^σ is a skew PBW extension of \mathbb{K} in n variables z_1, \dots, z_n defined by the relations $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$. So by Remark 4.4, $A^\sigma \cong B$ and therefore $Gr(A) \cong B$, i.e., A is a PBW deformation of B . \square

Note that if a skew PBW extension A is not constant then the Proposition 4.5 fails, indeed: the homogeneous version of A is the skew PBW extension B with relations $x_j x_i - c_{i,j} x_i x_j = 0$, where $c_{i,j}$ are the same as for A , but $Gr(A)$ is defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$ (see Theorem 2.5 and Proposition 2.4), so $Gr(A) \not\cong B$.

Let $T(V)/\langle P \rangle$ be a nonhomogeneous quadratic algebra. Take $R = p(P) \subseteq T^2(V)$ and consider the corresponding homogeneous quadratic algebra $A = T(V)/\langle R \rangle$. The main theorem of [6] establishes that if A is a homogeneous Koszul algebra then conditions (I) and (J) in Remark 4.4 imply that the algebra $T(V)/\langle P \rangle$ is a PBW deformation of A .

Proposition 4.6. *If A is a PBW deformation of a skew PBW extension B , then B is homogeneous Koszul.*

Proof. Let $A = \sigma(\mathbb{K})\langle x_1, \dots, x_n \rangle$ be a PBW deformation of B , then

$$A = \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle x_j x_i - c_{i,j} x_i x_j + k_0 + k_1 x_1 + \dots + k_n x_n \rangle,$$

with $c_{i,j} \in \mathbb{K} \setminus \{0\}$, $k_l \in \mathbb{K}$, $1 \leq i, j \leq n$, $0 \leq l \leq n$ and $B \cong \mathbb{K}\langle x_1, \dots, x_n \rangle / \langle x_j x_i - c_{i,j} x_i x_j \rangle$. Then B is a semi-commutative skew PBW extension of \mathbb{K} , and by Corollary 3.13, we conclude that B is homogeneous Koszul. \square

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References

- [1] J.P. Acosta, C. Chaparro, O. Lezama, I. Ojeda and C. Venegas, Ore and Goldie theorems for skew *PBW* extensions, *Asian-European J. Math.* **06** (2013) 1350061 [20 pages].
- [2] V. Artamonov, Derivations of Skew *PBW*-Extensions, *Commun. Math. Stat.* **3**(4) (2015) 449-457.
- [3] J. Backelin and R. Fröberg, Koszul algebras, Veronese subrings and rings with linear resolutions, *Rev. Roumaine Math. Pures Appl.* **30**(2) (1985) 85-97.
- [4] A. Beilinson, V. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, *J. Am. Math. Soc.* **9** (1996) 473-527.
- [5] R. Berger, Koszulity for nonquadratic algebras, *J. Algebra* **239** (2001) 705-734.
- [6] A. Braverman and D. Gaietsgory, Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type, *J. Algebra* **181** (1996) 315-328.
- [7] T. Cassidy and B. Shelton, Generalizing the notion of Koszul algebra, *Math. Z.* **260** (2008) 93-114.
- [8] C. Gallego and O. Lezama, Gröbner bases for ideals of σ -*PBW* extensions, *Comm. Algebra* **39**(1) (2011) 50-75.
- [9] C. Gallego and O. Lezama, d -Hermite rings and skew *PBW* extensions, *São Paulo J. Math. Sci.* **10**(1) (2016) 60-72.
- [10] H. Jiménez and O. Lezama, Gröbner bases for modules over σ – *PBW* extensions, *Acta Math. Academiae Paedagogicae Nyíregyháziensis* **32**(1) (2016) 39-66.
- [11] O. Lezama, J. P. Acosta and A. Reyes, Prime ideals of skew *PBW* extensions, *Rev. Un. Mat. Argentina* **56**(2) (2015) 39-55.
- [12] O. Lezama and A. Reyes, Some homological properties of skew *PBW* extensions, *Comm. Algebra* **42** (2014) 1200-1230.
- [13] H. Li, *Gröbner Bases in Ring Theory*, World Scientific Publishing Company, 2012.
- [14] L. Li, A generalized Koszul theory and its application, *Trans. Amer. Math. Soc.* **366**(2) (2014) 931-977.
- [15] J-L. Loday and B. Vallette, *Algebraic Operads*, Grundlehren Math. Wiss, Vol. **346** (Springer, Heidelberg, 2012).

- [16] P. Nuss, L'homologie cyclique des algèbres enveloppantes des algèbres de Lie de dimension trois, *J. Pure Appl. Algebra* **73** (1991) 39-71.
- [17] A. Polishchuk and C. Positselski, *Quadratic Algebras*, Univ. Lecture Ser., Vol. **37** (Amer. Math. Soc., Providence, RI, 2005).
- [18] S. Priddy, Koszul resolutions, *Trans. Am. Math. Soc.* **152** (1970) 39-60.
- [19] A. Reyes, Gelfand-Kirillov dimension of skew PBW extensions, *Rev. Col. Mat.* **47**(1) (2013) 95-111.
- [20] A. Reyes, Uniform dimension over skew PBW extensions, *Rev. Col. Mat.* **48**(1) (2014) 79-96.
- [21] A. Reyes, Jacobson's conjecture and skew PBW extensions, *Rev. Integr. Temas Mat.* **32**(2) (2014) 139-152.
- [22] A. Reyes, Skew PBW extensions of Baer, quasi-Baer, p.p. and p.q.-rings, *Rev. Integr. Temas Mat.* **33**(2) (2015) 173-189.
- [23] D. Rogalski, An introduction to non-commutative projective algebraic geometry, *arXiv:1403.3065 [math.RA]*.
- [24] S.P. Smith, Some finite dimensional algebras related to elliptic curves, *Rep. Theory of Algebras and Related topics* CMS Conf. Proc. **19**, AMS (1996) 315-348.
- [25] H. Suárez, O. Lezama and A. Reyes, Some relations between N -Koszul, Artin-Schelter regular and Calabi-Yau algebras with skew PBW extensions, *Revista Ciencia en Desarrollo* **6**(2) (2015) 205-213.
- [26] C. Venegas, Automorphisms for skew PBW extensions and skew quantum polynomial rings, *Comm. Algebra*, **43**(5) (2015) 1877-1897.
- [27] D. Woodcock, Cohen-Macaulay complexes and Koszul rings, *J. Lond. Math. Soc.* **57** (1998) 398-410.